VISTA: VISualization of relation Topologies of Alternatives

Robert Susmaga[a], **Izabela Szczęch** [a,*] **and Dariusz Brzezinski**[a]

[a]Institute of Computing Science, Poznan University of Technology, Poland
ORCID (Robert Susmaga): https://orcid.org/0000-0001-6707-0394, ORCID (Izabela Szczęch): https://orcid.org/0000-0002-9655-4109, ORCID (Dariusz Brzezinski): https://orcid.org/0000-0001-9723-525X

Abstract. The paper introduces VISTA (VISualization of relation Topologies of Alternatives), a novel visualization method for investigating multi-criteria ranking methods. Decision makers often struggle to understand why different ranking methods yield different results, leading to uncertainty in method selection and difficulty in justifying their decisions. VISTA addresses this problem by providing a visual lens into the inner workings of these methods by visualizing the preferential relations (preference, indifference, and incomparability) that these methods generate for pairs of alternatives. VISTA treats ranking algorithms as black boxes and queries them about the relations for pairs of alternatives. The resulting visualizations, called *vistas*, use color coding to represent the different relations. The paper demonstrates VISTA's utility by analyzing eight well-established ranking methods representing both functional and relational paradigms. The analysis reveals distinct patterns in how different methods handle preferential relations, exposing previously unrealized irregularities and inconsistencies. VISTA offers valuable insights for decision analysts and researchers by making the abstract inner workings of ranking methods visually interpretable and comparable.

1 Introduction

Being a sub-discipline of operations research, Multi-Criteria Decision Analysis (MCDA) aims to support decision makers in solving problems that involve real-world objects (alternatives) evaluated on multiple conflicting criteria. Often, this entails selecting the most preferred objects (also referred to as the α *problematic*), assigning them to preference classes (the β *problematic*), or ranking them from the most preferred to the least preferred (the γ *problematic*). In simplest cases, the ranking has the form of a total pre-order, although extensions of this structure (mainly involving incomparability) are also considered. For an extended overview of MCDA methods, models, and frameworks, see recent reviews [2, 10, 3, 8, 6, 16].

Among these three problematics, ranking methods have gained particular prominence due to their widespread application in high-stake domains that impact societies, including sustainable energy planning [1], logistics [4], manufacturing [20, 24], marketing [22], sustainable development [13], and engineering [11]. Within the extensive literature on ranking methods, two standard, predominantly popular paradigms have emerged: a *functional paradigm* (represented by, e.g., TOPSIS [9], SAW [9]) and a *relational paradigm* (represented by, e.g., the ELECTRE family of methods [14], PROMETHEE I/II [5]).

An important aspect in all those ranking methods is that they are intended to take into account the subjective preferences of individual decision makers, expressed as, e.g., criteria weights, thresholds like indifference or preference thresholds. This makes those methods 'unsupervised' in the sense that their results lack the final 'objective truth', further leading to difficulties in analyzing and comparing the ranking methods.

In this paper, we aim to address this difficulty by proposing an approach, called VISTA (VISualization of relation Topologies of Alternatives), for visualizing ranking methods. The approach is based on our observation that, regardless of the represented paradigm and the used method itself, all considered rankings may be expressed in terms of what is commonly referred to as *preferential relations*, i.e., three popular relations between alternatives: *preference*, *indifference*, and *incomparability*. The proposed approach visualizes those relations generated by particular methods for pairs of alternatives coming from a purposefully designed exhaustive dataset. By ensuring that the dataset is exhaustive, our approach provides general, dataset-independent results and conclusions. In particular, it can be used to reveal unrealized irregularities and inconsistencies in the inner workings of the visualized methods.

The remainder of this paper is organized as follows. In Section 2, we review the fundamental concepts of multi-criteria decision analysis that form the basis of our visual analysis. Section 3 formally introduces the VISTA visualization method and provides an illustrative example of its use. In Section 4, we showcase the method's utility through an in-depth analysis of eight prominent MCDA ranking methods, highlighting their distinct behaviors. Finally, Section 5 discusses the implications and limitations of our approach, whereas Section 6 concludes the paper with a summary of our contributions and directions for future research.

2 Preliminaries

2.1 Alternatives in the Criterion Space

In MCDA, the real-world objects under consideration are commonly referred to as *alternatives*. In this paper, the set of all possible alternatives will be denoted by $\mathbb{A}$ and assumed to satisfy $\mathbb{A} \neq \varnothing$. What is often considered in practice is $\mathcal{A} \subseteq \mathbb{A}$, a non-empty, finite subset of $\mathbb{A}$. The descriptions of alternatives consist of values of some predefined attributes. An attribute is thus a function that assigns a given

* Corresponding Author. Email: iszczech@cs.put.poznan.pl.

alternative a particular value from a pre-defined set of values, referred to as the domain of this attribute, and assumed here to be a non-degenerate real-valued interval. The resulting, multi-dimensional descriptions are usually represented as vectors. The attributes employed in MCDA, namely attributes with domains ordered according to preference in a weakly monotonic fashion, are referred to as *criteria*. In this paper, it is consistently assumed that *all* attributes are criteria, the set of which will be denoted by $\mathbb{K}$ and assumed to satisfy $\mathbb{K} \neq \varnothing$.

If $\mathcal{K} \in \mathbb{K}$, then its domain, which is a real-valued interval $\mathcal{V}$, is bounded by two values: the least preferred (denoted by v_*) and the most preferred (denoted by v^*). Of course, the values of v_* and v^* (and thus the resulting intervals) may differ for different criteria. Additionally, according to the monotonicity type of the ordering of its domain, criteria may differ in their preference types. In particular, criterion $\mathcal{K} \in \mathbb{K}$:

- is referred to as of type 'gain' when its domain $\mathcal{V} = [v_*, v^*]$, and the preference of $v \in \mathcal{V}$ does not decrease with the increase of v,
- is referred to as of type 'cost' when its domain $\mathcal{V} = [v^*, v_*]$, and the preference of $v \in \mathcal{V}$ does not increase with the increase of v.

Assume $\mathcal{K} \subseteq \mathbb{K}$, $|\mathcal{K}| = n \geq 1$, be a set of criteria selected from $\mathbb{K}$ and consider the *criterion space* CS, i.e., the set of all possible vectors $[v_1, v_2, ..., v_n]$ such that $v_j \in \mathcal{V}_j$, where $\mathcal{V}_j$ for $j \in \{1, 2, ..., n\}$ is the domain of criterion $\mathcal{K}_j \in \mathcal{K}$. The criterion space CS is thus an n-dimensional hyperrectangle $\mathcal{V}_1 \times \mathcal{V}_2 \times ... \times \mathcal{V}_n$ with 2^n vertices of the form $[s_1, s_2, ..., s_n]$, where $s_j \in \{v_{j*}, v_j^*\}$. In particular, CS contains two vertices:

- $[v_1^*, v_2^*, ..., v_n^*]$, further denoted as I and referred to as the ideal point, the representation of all the ideal alternatives from $\mathbb{A}$,
- $[v_{1*}, v_{2*}, ..., v_{n*}]$, further denoted as A and referred to as the anti-ideal point, the representation of all anti-ideal alternatives from $\mathbb{A}$.

Additionally, CS contains the following point $[\frac{v_1^*+v_{1*}}{2}, \frac{v_2^*+v_{2*}}{2}, ..., \frac{v_n^*+v_{n*}}{2}]$, further referred to as the midpoint. Notice that the midpoint is located in the very middle of CS, which means that it is equidistant from all the vertices of CS, including I and A.

Typically, MCDA methods assign weights to the criteria to express their relative importance. While real-world MCDA often involves varying criteria weights, we will use equal weights in this study to isolate and clearly visualize the fundamental relational topologies generated by different ranking methods. This simplification is a deliberate design choice that will allow us to focus on the core mechanisms of the methods themselves, rather than the influence of specific weight settings. Future work will explore the impact of weighted criteria on VISTA visualizations.

2.2 Preferential Relations

In this paper, we examine methods that use the most popular set of preferential relations [15], namely ones that use the relations of *preference*, *indifference*, and *incomparability*, further denoted as P, I and R, respectively. The assumed properties of these relations are as follows:

P (preference): an order,
I (indifference): an equivalence,
R (incomparability): an unlikeness.

Assuming the set-theoretic point of view, we will treat these relations as subsets of pairs from $X \times X$, obtained from some universe X. Subsequently, given $Rel \in \{P, I, R\}$ we will write $(x, y) \in Rel$ to denote that elements enter (satisfy) relation Rel.[1] In this paper, we will use the above notation to discuss preferential relations between pairs of alternatives, e.g., given $a, b \in \mathbb{A}$, $(a, b) \in P$ means that alternative a is better (than b), while alternative b is worse (than a). Simultaneously, $(a, b) \in I$ means that alternatives a and b are (mutually) indifferent, while $(a, b) \in R$ that they are (mutually) incomparable. We will also make use of the fact that alternatives $a, b \in \mathbb{A}$ have vector representations in CS. These representations, denoted as $\mathbf{a}, \mathbf{b}$, will also be similarly used with the three relations, e.g., $(\mathbf{a}, \mathbf{b}) \in I$ will mean that vectors $\mathbf{a}$ and $\mathbf{b}$ are indifferent, which will be treated as equivalent to stating that alternatives a and b are indifferent.

The relations P, I and R are assumed to be mutually disjoint ($P \cap I = \varnothing$, $I \cap R = \varnothing$, $R \cap P = \varnothing$). Moreover, P, P^{-1}, I and R are expected to be exhaustive ($P \cup P^{-1} \cup I \cup R = X^2$). Finally, I is not empty given $X \neq \varnothing$ (because it is reflexive), but P (and similarly R) may be either empty or not empty.

Let $\mathbf{b} \in CS$, $\mathbf{b} \neq \mathbf{a}$. Now, by definition, no more than one of $(\mathbf{a}, \mathbf{b}) \in P$ and $(\mathbf{b}, \mathbf{a}) \in P$ will occur simultaneously (because P is asymmetric); $(\mathbf{a}, \mathbf{b}) \in I$ and $(\mathbf{b}, \mathbf{a}) \in I$ will occur simultaneously or none of them will occur (because I is symmetric); $(\mathbf{a}, \mathbf{b}) \in R$ and $(\mathbf{b}, \mathbf{a}) \in R$ will occur simultaneously or none of them will occur (because R is symmetric). Because of the above, the statement 'a and b are indifferent' is equivalent to both $(a, b) \in I$ and $(b, a) \in I$ (because I is symmetric). Similarly for R (which is also symmetric). Contrastingly, 'a is preferred over b' is equivalent to $(a, b) \in P$ and $(b, a) \notin P$ (because P is asymmetric).

2.3 Relations in Ranking Paradigms

There are two typical MCDA paradigms considered for the ranking problematic: functional and relational. To recall their basic notions, let $\mathbb{A}$ denote all possible alternatives with $a, b \in \mathbb{A}$.

The basic idea behind the functional paradigm is as follows: it defines a real-valued function $F : \mathbb{A} \to [0, 1]$ and uses its values, $F(a)$ and $F(b)$, to determine the actual relation between a and b in the following way:

$$\begin{aligned} &\text{if } F(a) > F(b), \text{ then } (a, b) \in P, \\ &\text{if } F(a) < F(b), \text{ then } (b, a) \in P, \\ &\text{if } F(a) = F(b), \text{ then } (a, b) \in I \wedge (b, a) \in I. \end{aligned} \tag{1}$$

What is also important here, given $F(a), F(b) \in [0, 1]$, either $F(a) > F(b)$, $F(a) = F(b)$, or $F(a) < F(b)$ holds, which means that no other possibility exists, and thus the incomparability relation R must remain empty.

Next, the basic idea behind the relational paradigm is as follows: it defines a real-valued relation $S : \mathbb{A} \times \mathbb{A} \to [0, 1]$ and uses its values, $S(a, b)$ and $S(b, a)$, together with some threshold value $\lambda \in [0, 1]$, to determine the actual relation between a and b in the following way:

$$\begin{aligned} &\text{if } S(a, b) \geq \lambda \wedge S(b, a) < \lambda, \text{ then } (a, b) \in P, \\ &\text{if } S(a, b) \geq \lambda \wedge S(b, a) \geq \lambda, \text{ then } (a, b) \in I \wedge (b, a) \in I, \\ &\text{if } S(a, b) < \lambda \wedge S(b, a) \geq \lambda, \text{ then } (b, a) \in P. \end{aligned} \tag{2}$$

What is important, given $S(a, b), S(b, a), \lambda \in [0, 1]$, the above list of situations does not exhaust all the possibilities. The remaining situation ($S(a, b) < \lambda$ and $S(b, a) < \lambda$) is what populates relation R.

[1] The fact that (x, y) enters a relation may, but need not, imply that (y, x) enters the same relation depending on whether the relation is symmetric, asymmetric, or neither.

The above descriptions of functional and relational paradigms demonstrate that both rely on the assignment of pairs of alternatives to relations P, I, or R. In the following section, we present a visualization method that uses this fact to explain certain properties of MCDA ranking methods.

3 The VISTA Method

In this paper, we put forward a method called *VISTA*: *VISualization of relation Topologies of Alternatives*. The method provides insightful visualizations (called *vistas*) for a ranking algorithm by presenting the relative position (topology) of preferential relations (P, I, R) for pairs of alternatives. The ranking algorithms are thus treated as black boxes that we query about the relation they perceive between a pair of alternatives. Those relations are then depicted as colors. The considered pairs of alternatives constitute an exhaustive, purposefully designed dataset by taking into account a chosen (reference) alternative and all points surrounding it. Throughout this paper, we will use the midpoint in criterion space as the reference alternative, whereas the paired alternatives surrounding it will come from a uniform sample of the criterion space CS.

3.1 *Formal Definition*

For a given ranking method M, let $\mathbb{V} : CS \times CS \to \{\succ, \prec, \equiv, \#\}$ be a function defined as follows:

- $\mathbb{V}(\mathbf{a}, \mathbf{b}) = \succ$ when M implies that $(\mathbf{a}, \mathbf{b}) \in P$ and $(\mathbf{b}, \mathbf{a}) \notin P$,
- $\mathbb{V}(\mathbf{a}, \mathbf{b}) = \prec$ when M implies that $(\mathbf{b}, \mathbf{a}) \in P$ and $(\mathbf{a}, \mathbf{b}) \notin P$,
- $\mathbb{V}(\mathbf{a}, \mathbf{b}) = \equiv$ when M implies that $(\mathbf{a}, \mathbf{b}) \in I$ and $(\mathbf{b}, \mathbf{a}) \in I$,
- $\mathbb{V}(\mathbf{a}, \mathbf{b}) = \#$ when M implies that $(\mathbf{a}, \mathbf{b}) \in R$ and $(\mathbf{b}, \mathbf{a}) \in R$.

The fact that $P \cup P^{-1} \cup I \cup R = X^2$ means that the conditions $(\mathbf{a}, \mathbf{b}) \in P$, $(\mathbf{b}, \mathbf{a}) \in P$, $(\mathbf{a}, \mathbf{b}) \in I \wedge (\mathbf{b}, \mathbf{a}) \in I$ and $(\mathbf{a}, \mathbf{b}) \in R \wedge (\mathbf{b}, \mathbf{a}) \in R$ are exhaustive (at least one will always hold) and mutually exclusive (no more than one will ever hold) for every $(\mathbf{a}, \mathbf{b}) \in CS \times CS$. As a result, function $\mathbb{V}$ is well-defined.

Vistas will be visualizations composed of points in criterion space CS colored according to function $\mathbb{V}(\mathbf{a}, \mathbf{b})$. To make such visualizations practical, we set CS to two criteria of type gain, ranging from $v_* = 0$ to $v^* = 1$. Additionally, we propose to fix argument $\mathbf{a}$ to a vector representation of a user-specified *reference alternative* and provide $\mathbf{b}$ by sampling the *neighborhood* of $\mathbf{a}$ in criterion space. Throughout this paper, we will visualize 2D vistas using the midpoint of CS as the reference alternative. As the neighborhood, we will analyze a uniform sample of the entire criterion space. Points for which $\mathbb{V}(\mathbf{a}, \mathbf{b}) = \succ$ will be colored red and referred to as worse than the reference alternative. Analogously, when $\mathbb{V}(\mathbf{a}, \mathbf{b}) = \prec$ points will be colored green and referred to as better than the reference alternative. Similarly, we will use yellow to show $\mathbb{V}(\mathbf{a}, \mathbf{b}) = \equiv$ (indifference) and blue to show $\mathbb{V}(\mathbf{a}, \mathbf{b}) = \#$ (incomparability). An example vista is presented in Fig. 1.

3.2 *Illustrative example*

To better understand how the VISTA approach works, let us consider a hypothetical ranking method based on a simple unweighted mean (SUM). If the difference in the mean between two alternatives does not exceed a user-defined rating indifference threshold, then those alternatives are placed at the same ranking position (they are indifferent); otherwise, the one with the higher mean is better (placed at a

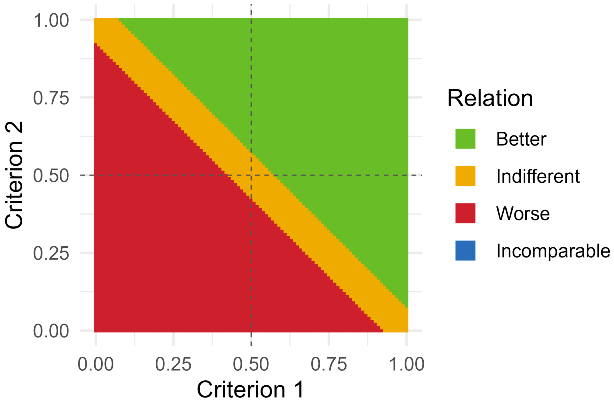

Figure 1. Vista for the SUM-based method (Eq. 3) for two criteria and the rating indifference threshold $\delta = 0.05$. Observe that there are no points perceived by the method as incomparable with the midpoint.

higher ranking position). This hypothetical ranking method is, thus, a simple representative of a functional paradigm that admits indifference and preference relations but not incomparability.

More formally, given representations $\mathbf{a}, \mathbf{b} \subseteq CS$ of alternatives, the SUM-based method ranks alternatives according to:

$$\begin{aligned} \mathbf{a} \succ \mathbf{b} &\iff F(\mathbf{a}) > F(\mathbf{b}) + \delta, \\ \mathbf{a} \equiv \mathbf{b} &\iff |F(\mathbf{b}) - F(\mathbf{a})| \leq \delta, \\ \mathbf{a} \prec \mathbf{b} &\iff F(\mathbf{b}) > F(\mathbf{a}) + \delta, \end{aligned} \quad (3)$$

where $F(\mathbf{x}) = mean(\mathbf{x}) = \frac{x_1+x_2}{2}$ for a vector $\mathbf{x} = [x_1, x_2]$, and $\delta \geq 0$ is the rating indifference threshold. Notice that Eq. 3 is clearly an instance of Eq. 1 (but rendered with a symbol-based notation).

A vista for this method is presented in Fig. 1. Each point in the plane represents a vector, which corresponds to a hypothetical alternative. The color of each point represents the relation implied by the ranking method between the reference alternative and an alternative represented by the point itself. For our example ranking method, the vista has only green, yellow, and red points, as there are no cases of incomparability (blue). The black dashed lines are shown to delineate the position of the midpoint (the reference alternative).

Let us consider the midpoint as our starting position for discussing the vista in Fig. 1. Moving along the horizontal dashed line to the right or left, we move to points with higher or smaller values of Criterion 1, respectively. The value of Criterion 2 for those points remains the same as for the midpoint. Analogously, when moving up or down the vertical dashed line, we keep the value of Criterion 1 unchanged while the values of Criterion 2 increase or decrease, respectively. Therefore, points on the same horizontal/vertical line can be interpreted as alternatives differing on only one of the criteria.

Now, let us consider traversing the main diagonal, i.e., the line connecting vertices $[0, 0]$ (anti-ideal point) and $[1, 1]$ (ideal point) in CS. Starting from the midpoint and moving towards the vertex $[1, 1]$, we move to points that proportionally increase both criteria. This reflects the increase of the mean criterion value for points located closer to the vertex $[1, 1]$. On the other hand, when moving towards the vertex $[0, 0]$, the points decrease their values on both criteria compared to the reference point (the mean decreases). The change in the mean as we traverse the main diagonal towards vertex $[1, 1]$ is reflected by the color changing from yellow in the middle to green, and analogously from yellow to red when approaching vertex $[0, 0]$. When the difference in mean does not exceed the rating indifference threshold δ, the color remains yellow.

Complementary to traversing the main diagonal is moving along the line perpendicular to it, i.e., the line connecting vertices $[0,1]$ and $[1,0]$. It shall be referred to as counter-diagonal. In those cases, a change in both criteria is observed, but it does not result in a change in the mean value. In other words, the proportions between the criteria values change making the values more and more diversified as the points approach either of vertices: $[0,1]$ or $[1,0]$, but the mean is unchanged. In our example vista, moving along the counter-diagonal results in no color change as the same mean implies the same ranking position with respect to the SUM-based method and, hence, indifference of alternatives. This is all reflected in the topology of colors that features a characteristic yellow stripe in Fig. 1. The width of the stripe is implied by the user-defined rating indifference threshold δ. In our example, $\delta = 0.05$.

4 Analysis of popular ranking methods using VISTA

To illustrate the usefulness of the proposed visualization approach, an analysis of eight well-established MCDA ranking methods (SAW [9], TOPSIS [9], E-TOPSIS [18], ORESTE [21], PROMETHEE I [5], VIKOR [23], ELECTRE III [14], ELECTRE IIIc [14]) was performed using vistas. The analyzed methods were chosen to represent both the functional and relational paradigms; they are listed and described in more detail in Table 1.

Table 1. Selected ranking methods representing the functional paradigm ($\mathbb{FUN}$) and the relational paradigm ($\mathbb{REL}$). The parameters used in the methods are ϵ: scaling coefficient, α: weighting factor between local rankings and global preference structure, δ: rating indifference threshold, F: function selector, q: indifference threshold, p: preference threshold, s: strategy coefficient, v: veto threshold, λ: credibility index.

method	paradigm	parameters[†]
SAW	$\mathbb{FUN}$	$\delta = 0.05$
TOPSIS	$\mathbb{FUN}$	$\delta = 0.05$
E-TOPSIS	$\mathbb{FUN}$	$\epsilon = 1$, $\delta = 0.05$
ORESTE	$\mathbb{FUN}$	$\alpha = 0.5$
PROMETHEE I	$\mathbb{REL}$	$F = t_4$, $q = 0.1$, $p = 0.2$
VIKOR	$\mathbb{FUN}$	$s = 0.5$
ELECTRE III[‡]	$\mathbb{REL}$	$q = 0.1$, $p = 0.2$, $v = 0.8$
ELECTRE IIIc	$\mathbb{REL}$	$q = 0.1$, $p = 0.2$, $v = 0.8$, $\lambda = 0.6$

† These parameters do not include the input weights, which with every method were set to 1 for each criterion.
‡ ELECTRE III has also parameters α and β (main controls common to the two distillation procedures), which are somewhat unexpectedly hard-coded by the pyDecision procedures to the ubiquitous values $\alpha = -0.15$ and $\beta = 0.30$ [19].

For the purpose of preparing vistas, in case of SAW, TOPSIS, ORESTE, PROMETHEE I, VIKOR, ELECTRE III, ELECTRE IIIc we have used implementations from the pyDecision library [12]. For E-TOPSIS [17, 18] a custom implementation was used. The names ELECTRE III and ELECTRE IIIc denote results of two separate stages of the multi-stage method ELECTRE III. The partial order is referred to as ELECTRE III, and the credibility matrix is referred to as ELECTRE IIIc (technically, the credibility matrix is produced at an earlier stage of the method than the partial order). Method ELECTRE IIIc is interesting in that it represents a direct implementation of the relational paradigm – in this implementation, the entries $c(a,b)$ of the credibility matrix play the role of the real-valued relations $S(a,b)$. Applying these values against the threshold λ leads to the determination of preferential relations. Even though ELECTRE IIIc produces relations directly, this need not be the case with all the other, in particular functional, methods. In fact, these methods typically produce rankings (partial or total), from which the preferential relations between alternatives are inferred by considering the individual ranks of the considered alternatives.

The different parameters used with all the considered methods are enumerated in Table 1. These particular parameter values are just examples, and it must be kept in mind that other settings may result in more or less different topologies of colors. Figure 2 shows vistas of the eight selected ranking methods.

Despite the similarities between the way SAW and the SUM-based method are defined, the vista of SAW is not like the one presented in Fig. 1. The difference is that the yellow strip in SAW curves towards the ideal point when it departs from the diagonal, while that of SUM runs straight. The explanation of this discrepancy lies in the fact that even though both use the mean, SAW also applies what is commonly referred to as the data normalization procedure, which clearly influences the method's ultimate result.

Next, let us observe that despite the similarities between TOPSIS and E-TOPSIS, their relation topologies are fairly different. Both their yellow strips widen as they depart from the main diagonal, but that of TOPSIS simultaneously curves towards the ideal point, in which it resembles that of SAW. Again, this fact may be explained by the application of the same normalization procedure in both TOPSIS and SAW, but a different one in E-TOPSIS.

The vista of E-TOPSIS can be considered similar to that of the SUM-based method (Fig. 1). The only noticeable difference is that the yellow strip in E-TOPSIS widens as it departs from the main diagonal, while that of SUM retains constant width. The explanation of this fact may be found in [18], where it was shown that the relative closeness used in E-TOPSIS can be decomposed into two elements, one of which happens to be the mean.

The vista of ORESTE strictly follows the domination/non-domination cones[2] of the midpoint. It is thus characterized by relatively large yellow areas, coming in the form of two square blocks in counter-diagonal aligned locations. As such, the pairwise topology of ORESTE behaves strictly like the dominance-based model.

Now, despite the profound differences (extending up to the level of the paradigm) in the underlying methods, the vista of PROMETHEE I combines those of ORESTE and E-TOPSIS/TOPSIS. The yellow indifference areas of PROMETHEE I appear as square blocks in counter-diagonal aligned locations – a feature of ORESTE. On the other hand, these yellow areas also increase in size as they depart from the main diagonal – a feature of E-TOPSIS/TOPSIS. The block-like topology of indifference is clearly triggered by the actual values of parameters, according to which a 3-level step function (as required by $F = t_4$) is applied to the criteria. As a result, three square blocks appear on either side of the midpoint (notice that all the blocks adjacent to the midpoint are merged into a single central block).

Again, despite the paradigm-reaching differences between the methods, the vista of VIKOR seems to be a fine-tuned version of that of PROMETHEE I – its yellow areas come in the form of square

[2] The dominated cone of point X will denote the set of all points that are dominated by X, while the dominating cone of point X will denote the set of all points that dominate X. The dominated and the dominating cones are collectively referred to as the domination cones, while all the remaining ones are referred to as the non-domination cones. In a plane (i.e. in two dimensions) there are two domination cones, and two non-domination cones, all of which are generally delineated by a pair of straight lines (a horizontal and a vertical) that cross in X.

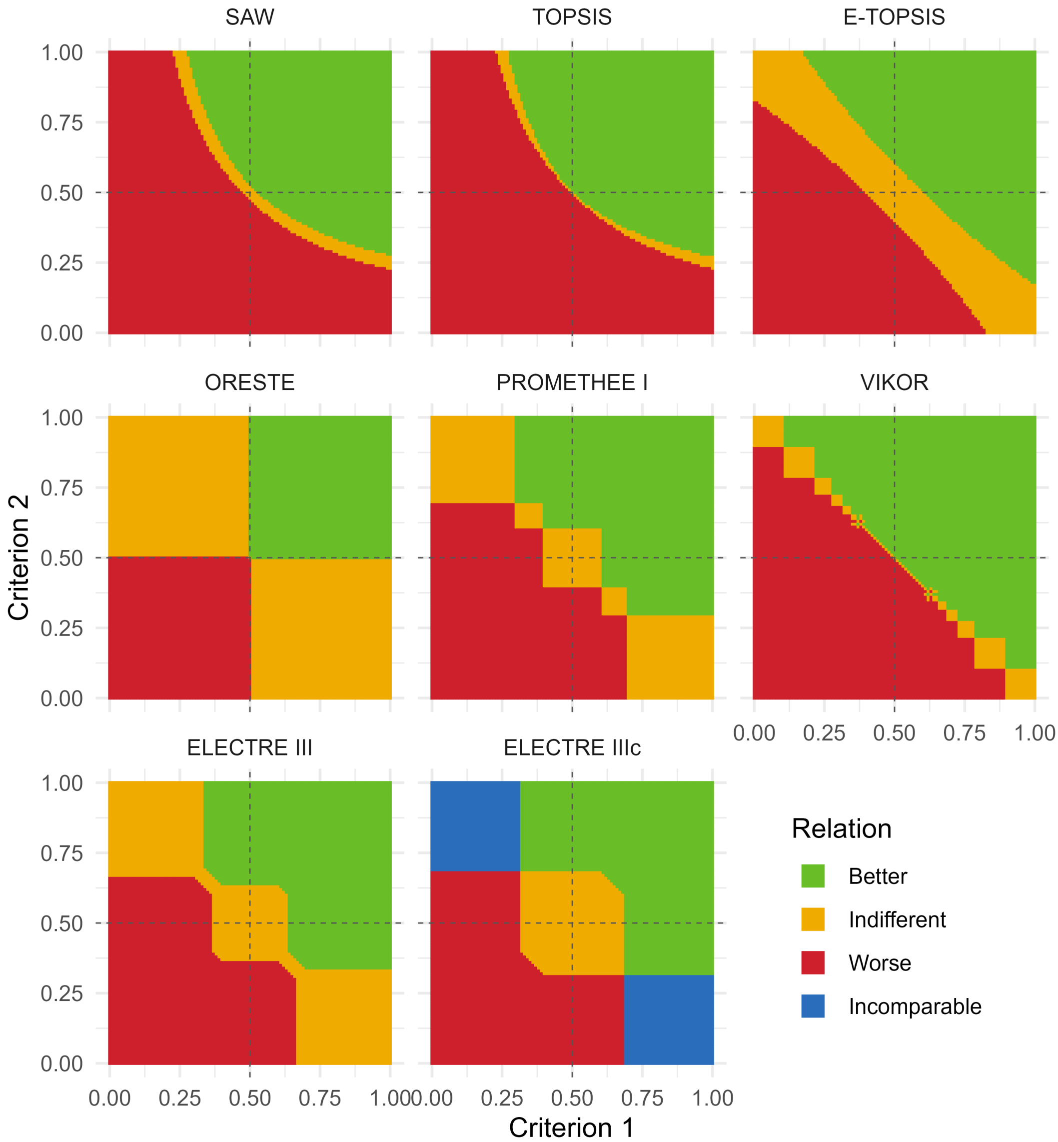

Figure 2. 2D vistas for eight established MCDA ranking methods; see Table 1 for value sets of applied parameters.

blocks (though much more numerous) in counter-diagonally aligned locations, the size of which increases gradually as they depart from the main diagonal. Even though VIKOR represents the functional paradigm, its vista is in a sense much like those generated by methods that represent the relational paradigm.

Next, the vista of ELECTRE III somewhat follows those of PROMETHEE I and TOPSIS – its yellow areas come in the form of a union of: three counter-diagonally aligned roughly square blocks of slightly increasing sizes, and a thin, also counter-diagonally aligned, strip of constant width. The presence of the strip, which may be viewed as having come from some form of 'erosion' of the middle (initially larger) block, gives ELECTRE III some appearance of a method representing the functional paradigm.

Finally, the vista of ELECTRE IIIc, basically similar to that of ELECTRE III, contains a distinct feature, i.e., blue areas that occupy the utmost two of the three blocks. These areas denote cases of incomparability, which happen to occur in the results of ELECTRE IIIc. The structural similarity of the relation topologies produced by ELECTRE III and ELECTRE IIIc is, in this case, caused by the identical set of parameter values applied to the common parameters of these methods.

The individual visualizations obtained with the VISTA method actually call for some form of their generalization/formalization. Meeting this challenge, we formulated the following tentative postulates, expressed in terms of the relation topologies, understood as the *expected properties* of the ranking methods.

(1) no red/green points reside in the dominating/dominated cone of the midpoint,
(2) the midpoint is yellow,
(3) the color is subject to at most one change (to green/red) when moving from the midpoint along the main diagonal (towards the ideal/anti-ideal point),
(4) all colors manifest axial symmetry around the main diagonal,
(5) the color is subject to at most one change when moving from the midpoint along any direction outwards,
(6) all colors manifest axial symmetry around the counter-diagonal,
(7) the areas occupied by colors green and red are equal.

First of all, the postulates are formulated in terms of simplified "planar squares with midpoints" environments, i.e. for cases in which the CS is a planar (implying $n = 2$) square (implying equal domains) and the reference point is located in its very middle; in other words, they are formulated for 2D vistas. This simplification is well justified as this is exactly the character of all the vistas presented in this paper. Although more general vistas and thus more general postulates are certainly conceivable, e.g. for $n > 2$ they would involve cubes ($n = 3$) or hypercubes ($n > 3$). Such an approach would complicate things, as not all used notions naturally generalize for $n > 2$ (e.g. a 'orthogonal subspace' and 'hyper-volume' should be used instead of 'counter-diagonal' and 'area', respectively).

Next, the postulates are generally enumerated in the expectancy-decreasing order, which means that while the initial ones (especially (1)..(3)) constitute what may be referred to as 'must do's' (a method that generates vista not satisfying (1) could clearly be treated as flawed), the trailing ones (especially (5)..(7)) need not be always fully satisfied (a method that generates vista not satisfying (7) could be vindicated without much trouble). The nuanced character of this ordering is clearly demonstrated in a more detailed interpretations of the postulates (below).

Finally, the postulates are by no means mutually independent, as e.g. (6) implies (7) and (5) implies (3), while e.g. (1), (2) and (3) are clearly related (though with no implications involved).

For a sample of a more detailed discussion, consider the interpretation of the (interdependent) postulates (1), (2), (3) and (5). Postulate (1) seems most obvious, as it simply restates the dominance principle (a dominated vector cannot be treated as better than the dominating one, and a dominating vector cannot be treated as worse than the dominated one). Similarly, postulate (2) seems obvious, as it simply restates that the midpoint (vector $[0.5, 0.5]$), should be treated as indifferent from itself (a fairly straightforward conclusion).

Now, consider (3), a less obvious one, which (in version one) states that when moving from the midpoint along to the main diagonal towards the ideal (vertex $[1, 1]$) the color may only change once (to green). Start with two allowed situations. First: the color does not change. This would mean that the method treats all the vectors between the midpoint and the ideal as indifferent with the midpoint. This is certainly allowed. Next: the color changes once (to green). Let the change occur in the breakpoint (vector $[b, b]$, where $0.5 < b \leq 1$). This would mean that the method treats all the vectors between the midpoint up to the breakpoint as indifferent with the midpoint, and all the vectors between the breakpoint and the ideal as preferred over the midpoint. This is also allowed.

As it can be easily guessed, all other situations are forbidden or at least undesirable, which is best justified by counterexamples. Start with situations in which the color changes once, but to one of the two remaining colors. First: red. This means that the method treats all the vectors between the breakpoint and the ideal as worse than the midpoint. This would clearly violate postulate (1) and is thus categorically forbidden. Next: blue. This means that the method treats all the vectors between the breakpoint and the ideal as incomparable with the midpoint. This would be strongly undesirable (though not categorically forbidden) owing to the special role of the ideal.

Similar reasoning applies in situations in which the color changes more than once. First of all, situations that are forbidden/undesirable in the result of the first change remain forbidden/undesirable no matter what the further changes are. Consider thus a first, allowed change after which comes a second change. Let the second change occur in the second breakpoint (vector $[s, s]$, where $b < s \leq 1$). The change can be evaluated by the relation assigned to all the vectors between the second breakpoint and the ideal, in the same way in which the first change was evaluated by the relation assigned to all the vectors between the breakpoint and the ideal. What emerges from inductively considering all such cases is that the second change may only deteriorate the evaluation. In result, postulate (3) in version one (about the color's allowed one change to green when moving towards the ideal) may be regarded as well justified.

By analogy, the same reasoning justifies version two of this postulate (about the color's allowed one change to red when moving towards the anti-ideal), which simply considers a movement in the opposite direction. Interestingly, analogous reasoning also justifies the more general postulate (5), which considers a movement in any direction (thus (3) is a special, though more desired, case of (5)).

Vistas clearly support swift verification of whether a method possesses a given postulate or not. The results of this verification for particular vistas generated the eight considered MCDA ranking methods are gathered in Table 2.

Though we do not intend to conclusively favor one ranking method over another on the basis of those properties, this verification can give the decision maker a valuable insight into the behavior of these methods. This can further help in choosing a method for the task at hand in an informed manner. For example, compare the vista of the SUM-based method (Fig. 1) to those of SAW and TOPSIS (Fig. 2). While

Table 2. Satisfiability of the postulates by the presented vistas of the considered MCDA ranking methods.

Ranking method	Preference relation topology expectation						
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
SAW	+	+	+	+	+	−	−
TOPSIS	+	+	+	+	+	−	−
E-TOPSIS	+	+	+	+	+	+	+
ORESTE	+	+	+	+	+	+	+
PROMETHEE I	+	+	+	+	−	+	+
VIKOR	+	+	+	+	−	+	+
ELECTRE III	+	+	+	+	−	+	+
ELECTRE IIIc	+	+	+	+	−	+	+

the yellow strip in the first is straight all the way, the same strips in the other methods are different, becoming slightly wider (TOPSIS) and curving towards the ideal point (both SAW and TOPSIS) as they depart from the diagonal. Recalling that the vista is always generated for the midpoint, consider vectors $\mathbf{b}_{01} = [0.0, 1.0]$, and $\mathbf{b}_{10} = [1.0, 0.0]$: while $\mathbf{b}_{01}$ and $\mathbf{b}_{10}$ are included in the yellow strip of SUM-based method, this is not the case with SAW and TOPSIS, where they are included in the red region. Clearly the SUM-based method treats $\mathbf{b}_{01}$ and $\mathbf{b}_{10}$ as indifferent from midpoint, but SAW and TOPSIS treat the same vectors as worse than the midpoint (even though $mean(\mathbf{b}_{01}) = mean(\mathbf{b}_{10}) = 0.5$, which is the same as the mean of the midpoint). But this is indicative of how the methods work: as opposed to the SUM-based method, SAW and TOPSIS have a slight, but clear and systematic bias towards vectors with more balanced elements. As a result, you need, e.g., vector $\mathbf{c} = [0.2, 1.0]$ (imbalanced elements) with $mean(\mathbf{c}) = 0.6 > 0.5$ to be treated as indifferent from the midpoint (balanced elements).

Now, imagine a similar analysis for VIKOR, the vista of which clearly shows how difficult it is to generally predict what a given vector with a mean close to 0.5 will be treated like: indifferent, worse, or better than the midpoint. While it is thus clear that VIKOR has some bias (just like SAW and TOPSIS), the character of this bias is hard to describe, making this method potentially less predictable and thus much more difficult to handle.

5 Discussion

VISTA offers insightful analysis of ranking methods, producing visualizations, called vistas, that reveal their inner-workings by depicting relations occurring between pairs of alternatives. We have shown that popular representatives of both functional and relational paradigms can be analyzed and compared using vistas.

Apart from restating the main characteristic of the proposed visualization method, it is worth listing its potential limitations. Notice that VISTA is mainly concerned with obtaining the relations from the methods. After the set of systematically sampled relations for a given reference point and a criterion space is ready, its topology is shown 'as is'. Because such visualizations with more than two criteria are hard or even impossible to construct, practical ones are limited to two-dimensional cases only. This is because the dataset in VISTA (i.e. the criterion space) is not submitted to any dimensionality reducing transformations (these could possibly complicate the interpretation), which constitute the core of extremely numerous dimensionality-reduction based visualization techniques, see e.g. [7] just for linear ones.

Nevertheless, even the included 2D vistas are informative enough to draw very insightful conclusions regarding the underlying methods. The fact that basically all points are considered and that they come from a uniform sample of the criterion space allows for general, data-set-independent conclusions. This is because the vistas present all parts of the criterion space, including parts that might never be analyzed when real-life datasets are used.

The VISTA approach visualizes relations perceived by the analyzed ranking method between a pair of alternatives. However, not all of such pairs can be considered owing to the resulting growth of dimensionality. To reduce this growth, we fix the first element of the pair to a user-specified reference alternative. This is necessary to make the visualizations possible, even though it can be regarded as a restriction of the general methodology. In the future, multiple reference points (and multiple vistas) may be considered.

Finally, the color of the presented point is obtained by querying a ranking method about the relation it perceives between a pair of alternatives. Thus, the ranking method receives as input a pair of vectors representing two alternatives: the reference one and the analyzed one. Interestingly, some of the relational ranking methods may practically produce no cases of incomparability when presented with only two alternatives. As a result, their vistas do not contain any blue points despite the fact that the methods are generally known for admitting incomparability. This, however, should not be regarded as a limitation of the VISTA, but a rather surprising, input-dependent feature of the ranking methods themselves. In the future, we plan to investigate this phenomenon also on visual basis.

6 Conclusions

The paper puts forward VISTA (VISualization of relation Topologies of Alternatives), a novel visualization method for analyzing ranking methods. The approach addresses a valid challenge in MCDA – the difficulty of interpreting and comparing ranking methods due to their subjective nature and general lack of objective truth. By visualizing the preferential relations that the ranking methods perceive between alternatives, VISTA supports their efficient comparison and analysis.

The key contributions of this work are threefold. Firstly, a novel visualization approach was proposed. VISTA treats ranking algorithms as black boxes and queries them about preferential relations, making their inner-working observable through color-coded vistas. Secondly, an analysis of eight established MCDA methods was conducted, giving the decision maker a valuable insight into the behavior of these methods. Finally, general properties for evaluation of method quality were suggested.

The exhaustive dataset approach ensures that conclusions drawn from vistas are general and dataset-independent, making the method particularly valuable for understanding fundamental method behavior rather than their performance on specific datasets. This characteristic positions VISTA as a complementary tool to traditional benchmarking approaches to MCDA ranking methods. Future works in this area may thus focus on providing extended visual analyses of these methods, e.g. with changing weights and/or parameter values. Moreover, the established framework opens also entirely new avenues for analyzing, understanding and improving various multi-criteria decision support systems. In particular, further investigations may focus on extending the visualization approach to methods in other MCDA problematics, e.g. in multi-criteria sorting.

Acknowledgements

This research was partly funded by the National Science Centre, Poland, grants number 2022/47/D/ST6/01770 and 2023/51/B/ST6/00545.

References

[1] C. Becchio, M. Bottero, S. Corgnati, F. Dell'Anna, and G. Vergerio. Energy audit and multi-criteria decision analysis to identify sustainable strategies in the university campuses: Application to Politecnico di Torino. In C. Bevilacqua, F. Calabrò, and L. Della Spina, editors, *New Metropolitan Perspectives*, pages 1187–1197, Cham, 2021. Springer International Publishing. ISBN 978-3-030-48279-4.

[2] V. Belton and T. Stewart. *Multiple criteria decision analysis: an integrated approach*. Springer, 2002.

[3] R. Bisdorff, L. Dias, P. Meyer, V. Mousseau, and M. Pirlot. *Evaluation and Decision Models with Multiple Criteria*. Springer, 2015.

[4] E. Bottani and A. Rizzi. A fuzzy TOPSIS methodology to support outsourcing of logistics services. *Supply Chain Management*, 11(4):294–308, 2006.

[5] J. Brans. L'ingénièrie de la décision; Elaboration d'instruments d'aide à la décision. La méthode PROMETHEE. In R. Nadeau and M. Landry, editors, *L'aide à la décision: Nature, Instruments et Perspectives d'Avenir*, pages 183–213, Québec, Canada, 1982. Presses de l'Université Laval.

[6] M. Cinelli, M. Kadziński, G. Miebs, M. Gonzalez, and R. Słowiński. Recommending multiple criteria decision analysis methods with a new taxonomy-based decision support system. *European Journal of Operational Research*, 302(2):633–651, 2022.

[7] J. P. Cunningham and Z. Ghahramani. Linear dimensionality reduction: Survey, insights, and generalizations. *Journal of Machine Learning Research*, 16:859–2900, 2015.

[8] S. Greco, M. Ehrgott, and J. Figueira. *Multiple Criteria Decision Analysis: State of the Art Surveys*. Springer, 2016.

[9] C. L. Hwang and K. Yoon. *Multiple Attribute Decision Making: Methods and Applications*. Springer-Verlag, Berlin Heidelberg, 1981.

[10] A. Ishizaka and P. Nemery. *Multi-criteria Decision Analysis: Methods and Software*. Wiley, 2013.

[11] S.-S. Lin, A. Zhou, and S.-L. Shen. Safety assessment of excavation system via TOPSIS-based MCDM modelling in fuzzy environment. *Applied Soft Computing*, 138:110206, 2023.

[12] V. Pereira, M. P. Basilio, and C. H. T. S. H. T. Santos. Enhancing decision analysis with a large language model: pyDecision a comprehensive library of MCDA methods in Python. *arXiv:2404.06370*, 2024.

[13] M. Piwowarski, D. Miłaszewicz, M. Łatuszyńska, M. Borawski, and K. Nermend. TOPSIS and VIKOR methods in study of sustainable development in the EU countries. *Procedia Computer Science*, 126: 1683–1692, 2018.

[14] B. Roy. The outranking approach and the foundations of ELECTRE methods. *Theory and Decision*, 31:49–73, 1991.

[15] B. Roy. *Multicriteria Methodology for Decision Aiding*. Kluwer Academic, Berlin, 1996.

[16] W. Sałabun, J. Wątróbski, and A. Shekhovtsov. Are MCDA methods benchmarkable? A comparative study of TOPSIS, VIKOR, COPRAS, and PROMETHEE II methods. *Symmetry*, 12(9), 2020. ISSN 2073-8994. doi: 10.3390/sym12091549. URL https://www.mdpi.com/2073-8994/12/9/1549.

[17] R. Susmaga and I. Szczęch. Elliptic generalizations of TOPSIS. In *Multi-Objective Decision Making (MODeM) Workshop Kraków Poland 1-10-2023*, 2023. URL https://modem2023.vub.ac.be/papers/MODeM2023_paper_5.pdf.

[18] I. Szczęch and R. Susmaga. Utility-inspired generalizations of TOPSIS. *Neural Computing and Applications*, 2025. doi: 10.1007/s00521-025-11238-x.

[19] D. Vallee and P. Zielniewicz. *ELECTRE III-IV, version 3.x : aspects méthodologiques*. Document du Lamsade 85, 1994.

[20] W.-P. Wang. Toward developing agility evaluation of mass customization systems using 2-tuple linguistic computing. *Expert Systems with Applications*, 36(2):3439–3447, 2009.

[21] M. Yerlikaya, K. Yildiz, and B. Keskin. Solution proposal for completed preference structure in ORESTE method. *Scientific Reports*, 13(4754), 2023.

[22] X. Yu, S. Guo, J. Guo, and X. Huang. Rank B2C e-commerce websites in e-alliance based on AHP and fuzzy TOPSIS. *Expert Systems with Applications*, 38(4):3550–3557, 2011.

[23] E. Zavadskas, A. Zakarevičius, and J. Antucheviciene. Evaluation of ranking accuracy in multi-criteria decisions. *Informatica, Lith. Acad. Sci.*, 17:601–618, 01 2006.

[24] Z. Zhang, H. Jiang, T. Shao, and Q. Shao. Understanding the selection of intelligent engineering B2B platform in China through the fuzzy DANP and TOPSIS techniques: A multi-study analysis. *Applied Soft Computing*, 141:110277, 2023.